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Adaptive particle distribution for Smoothed Particle Hydrodynamics

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SUMMARY

A framework for adaptively inserting and removing particles with smoothed particle hydrodynamics (SPH) has been developed. A number of SPH variants were examined for use in an adaptive method. A minimum of linear consistency in the method has proven critical. Algorithms for particle placement and reassignment are discussed and results for a shock tube problem are shown. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: smoothed particle hydrodynamics; adaptive particle distribution; consistency

1. INTRODUCTION

Smoothed particle hydrodynamics (SPH) is a meshless technique for simulations in computational fluid dynamics. It was originally developed for astrophysics by Lucy [1] and Gingold and Monaghan [2], but has been adapted by a range of problems in fluid mechanics [3–5]. As a Lagrangian technique, it offers advantages in problems with multiple phases, complex geometries and moving boundaries.

The resolution for any particular simulation is dependent on the distribution of computational particles, and can be controlled to some degree by the particles' initial distribution. However, as the solution evolves, the particle distribution is dictated by the flow and local control of resolution is lost. In some applications, the distribution dictated by the flow leads to a high number of particles in locations where higher resolution is desired. To take advantage of this, smoothing lengths are made to vary from particle to particle. Shapiro *et al.* [6] extended this idea and allowed smoothing length to vary directionally.

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Another approach to improving control of local resolution is to adjust the particle distribution itself, as in conventional adaptive mesh refinement. Kitsionas and Whitworth [7] have implemented a particle-splitting method for astrophysics. Liu *et al*. [8] presented an approach for inserting particles in an Eulerian form of the reproducing kernel particle method for CFD applications. More recently, Liu [9] demonstrated a Lagrangian implementation of a meshless method for particle insertion, with the aid of a Delaunay triangulation.

This work presents a more general method for resolution adjustment using particle insertion and removal. Any number of particles can be inserted during the adjustment step with few restrictions on their placement. Also, particles can be removed where lower resolution can be accepted. The approach is exclusively Lagrangian and uses simple mechanisms for particle placement and re-assignment of smoothing length and particle volume.

2. THE SPH METHOD

SPH is based on a method of estimating function values and gradients when the function values are known for a set of disordered points. Following Monaghan [10], the value of an arbitrary function F at a particle i is approximated as

$$\langle F_i \rangle = \sum_j V_j F_j W_{ij} \tag{1}$$

where the kernel W is a function of the smoothing length h and the distance between particle i and its neighbour j. The fluid volume assigned to each particle, V, is usually expressed in terms of density and particle mass as

$$V_j = \frac{m_j}{\rho_j} \tag{2}$$

The gradient of F can be found by simply taking the gradient of the kernel

$$\langle \nabla F_i \rangle = \sum_j V_j F_j \nabla W_{ij} \tag{3}$$

The governing equations of fluid dynamics are modelled using Equation (3) to compute required gradients. More details on the basic SPH method can be found in References [4, 9, 10].

Typical SPH kernel functions satisfy the normalization condition that the integral of the kernel over its region of influence should equal unity [10]. However, this condition is not generally satisfied in the discrete summation form as applied in SPH:

$$\sum V_j W_{ij} \neq 1 \tag{4}$$

As a result, SPH cannot correctly estimate the values or gradients of a constant function. Belytschko [3] introduced the definition of consistency order from the finite element method as the order of polynomial that can be represented exactly. Basic SPH does not achieve 0 order consistency. The lack of consistency leads to significant error when the particle spacing is non-uniform. To develop an effective method for inserting and removing particles as a solution progresses, the formulation must be accurate when spacing inevitably becomes non-uniform.

3. MODIFIED SPH

Five modified SPH formulations with higher orders of consistency are now described. Each method is evaluated for its ability to reproduce polynomial test functions when particle spacing is non-uniform. This directly addresses the main difficulty of adaptive particle distribution.

In many cases, the gradient of a function F_i that is used in a conservation equation can be written in the form

$$\langle \nabla F_i \rangle = \sum_j V_j (F_j - F_i) \nabla W_{ij} \tag{5}$$

Then if F is constant, $F_i = F_j$, and the gradient will always be correctly evaluated as zero—regardless of particle spacing. In most SPH implementations, the energy and continuity equations already appear in this form. Following References [4, 11], the momentum equation can also be expressed in this way, though Bonet and Lok [11] caution against such a use.

Alternatively, the kernel function itself can be modified to achieve zero-order consistency. For example, the Shepard function [11, 12] shown here directly normalizes the kernel function to yield a new corrected kernel W_{ij}^0, which ensures that the normalization condition is satisfied.

$$\langle F_i \rangle = \frac{\sum_j V_j F_j W_{ij}}{\sum_j V_j W_{ij}} = \sum_j V_j F_j W_{ij}^0 \tag{6}$$

Kernel corrections can be extended to higher orders of consistency, as shown by Liu *et al.* [13, 14]. Here, the α and β terms make corrections that guarantee exactly correct evaluation of a linear function F

$$\langle F_i \rangle = \sum_j V_j F_j W_{ij} [\alpha + \beta \cdot (\mathbf{r}_i - \mathbf{r}_j)] \tag{7}$$

where $\mathbf{r}_i$ represents the location of particle i.

Another way to improve consistency for the Shepard function is to follow the mixed kernel and gradient correction proposed by Bonet and Lok [11]. A first-order correction $\mathbf{L}_i$ is applied to the gradient of the zero-order kernel from Equation (6):

$$\langle \nabla F_i \rangle = \sum_j V_j F_j \mathbf{L}_i \nabla W_{ij}^0, \quad \mathbf{L}_i = \left[\sum_j V_j \nabla W_{ij}^0 \otimes x_j \right]^{-1} \tag{8}$$

A final method uses a combination of the features from two previous methods. To achieve zero-order consistency, the conservation equations are written in the difference form (Equation (5)). The gradient correction $\mathbf{L}_i$ from (8) is then applied to obtain 1st order consistency.

The standard SPH formulation and the five modified forms were evaluated for their ability to accurately reproduce test functions and their gradients. The functions were represented in one dimension by a set of 51 particles. These particles were regularly spaced at an interval of $\Delta x = 1.0$ with the exception of a single point where the distance to its left and right nearest neighbours was $\Delta x/2$. To illustrate the errors associated with irregular spacing, a plot of the gradient calculation results for a quadratic test function is shown in Figure 1. Here, standard SPH is compared to the Shepard's function from Equation (6) (0 order consistent) and the Shepard's function with a gradient correction from Equation (8) (1st order consistent).

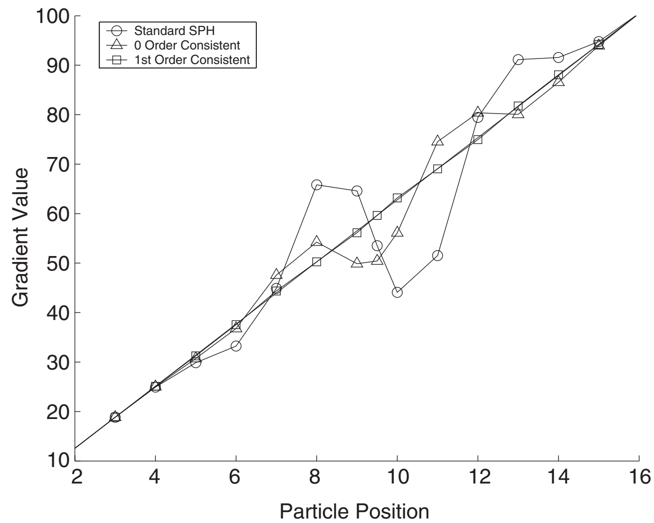

Figure 1. Gradient of a quadratic function near non-uniform spacing.

Table I. Total error from gradient calculations.

Method	Consistency	$\nabla F=0$	$\nabla F=3.14$	$\nabla F=6.28x$
Standard SPH	None	0.4	3.4	33.9
Difference form	0-order	0	0.5	9.6
Shepard's function	0-order	0	0.7	13.4
Linearly consistent kernel	1-order	0	0	1.2
Shepard's function + gradient correction	1-order	0	0	0.8
Difference form + gradient correction	1-order	0	0	0.8

The results of the gradient evaluation are shown in Table I, where error is calculated as the square root of the total squared difference between the SPH approximation and the exact gradient. As the order of consistency increases, so does the accuracy of the result near the irregular spacing. This is true even if the order of the polynomial exceeds the consistency order of the method. Methods of a similar order of consistency all yielded similar results, suggesting that consistency is a critical property in determining how well a method performs in the face of non-uniform spacing. In addition, consistency corrections can offer advantages such as improved boundary handling and, in the case of the gradient correction, a guaranteed conservation of angular momentum [11].

4. ADAPTIVE PARTICLE INSERTION AND REMOVAL

Once an appropriate SPH formulation has been established, the actual procedure used to add and remove particles must be defined. This is broken down into five steps:

(i) The criteria for adding and removing particles must be defined.
(ii) Particles must be added and removed according to the criteria.

(iii) Flow variables must be interpolated for newly added particles.
(iv) The smoothing lengths must be re-assigned to reflect the new distribution.
(v) The mass of each particle must be redistributed to reflect the new distribution.

There are many possible criteria for particle addition and removal (step (i)). In conventional mesh-based CFD, techniques for choosing locations of mesh refinement have become quite sophisticated [15]. The objective of the present work is to develop a robust and flexible framework for SPH in which particles can be inserted and removed according to any criterion. To generate results for demonstration in Section 5, a criterion based on velocity gradient has been chosen. Where local velocity gradients are high, particles are added; where local gradients in the solution are low, particles are removed, subject to upper and lower limits on particle spacing (and effectively, on smoothing length).

With the criteria set, addition and removal of particles can take place (step (ii)). For addition, a new particle is placed near an existing particle with a high velocity gradient. Its position is then iteratively adjusted to improve the local inter-particle spacing, giving a more even distribution. This step reduces error associated with uneven spacing, as discussed in Section 3. Particles identified for removal in step (i) are simply deleted from the calculations. The re-allocation of their fluid volume and particle mass is considered in steps (iv) and (v).

Now that new particle locations have been determined, a first-order consistent SPH kernel is used to interpolate flow variables at the new locations (step (iii)). Higher order consistent kernels could be used to improve accuracy, but this is balanced by computational cost.

In step (iv), smoothing lengths are reassigned to reflect the new particle distribution. A constant reference smoothing length, h_{ref}, is introduced here to assess the local spacing. The sum over neighbouring kernel values is calculated for each particle using this reference smoothing length, giving an approximate value for the volume represented by each particle. Specifically,

$$V_i^{\mathrm{ref}} = \frac{1}{\sum_j W(|\mathbf{r}_j - \mathbf{r}_i|, h_{\mathrm{ref}})} \tag{9}$$

This effectively relates the volume to the smoothed value of number density, ρ/m. The new smoothing length is then proportional to $(V_i^{\mathrm{ref}})^{1/d}$, where d is the number of dimensions.

Finally in step (v), mass is redistributed using each particle's fluid volume and the original density distribution. This is done by first replacing the reference smoothing length in Equation (9) with the local smoothing length. This defines the fluid volume of each particle based on inter-particle spacing, and the mass redistribution is performed with Equation (2) to restore the original density distribution. Consistency-corrected methods are critical here, allowing the original density profile of the fluid to be more accurately reproduced.

5. SHOCK TUBE RESULTS

Adaptive particle distribution was implemented with the various underlying SPH formulations from Section 3, and applied to compressible flow with the Riemann shock tube problem. The shock tube was implemented with an initial 4:1 pressure ratio and a Courant number of 0.3 (based on smoothing length). A snapshot of the resulting velocity profiles near the shock front is shown in Figure 2. Methods of similar orders of consistency yielded similar results, so the methods from Equations (6) and (8) are used here as representatives for 0 and 1 order

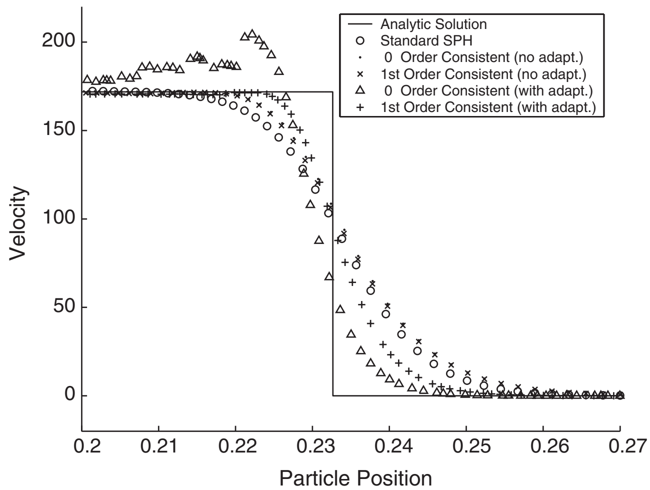

Figure 2. Instantaneous velocity profiles at shock front.

consistency corrections, respectively. The corrected methods applied without adaptive particle distribution showed little difference at the shock front but greater error at the expansion wave when compared to the standard SPH method. Errors in density were compared by calculating the absolute difference from the analytic solution, weighed by each particle's volume and normalized by the total mass. The corrected methods showed density errors of 0.74% and 0.71% for zero- and first-order consistency corrections respectively, compared to 0.51% for standard SPH.

When adaptive distribution is applied, the zero-order consistent method has difficulty accommodating particle addition and removal. Although the front of the shock is sharper, there are significant post-shock errors leading to a total density error of 0.94%. The first-order consistent method with adaptivity, however, improves the shock front resolution while maintaining the correct post-shock solution. The solution at the expansion wave is also improved, and the total density error is reduced to 0.32%. The total number of particles was kept at around 450—the same total number used in the non-adaptive cases.

As a first step towards multi-dimensional applications of the technique, simulations have been carried out for this nominally 1D shock tube flow with a finite width in the second dimension. In SPH, this problem does not reduce trivially to the 1D problem as it would in a mesh-based technique with a suitably flow-aligned grid. Improvement in shock capturing was achieved with adaptive particle distribution, as in the 1D case. The global density error, as defined above, was reduced from 1.8% for standard SPH to 0.67% for the adaptive case. The total number of particles was again kept similar for both cases at around 4400.

6. CONCLUSIONS

The ability to adaptively insert and remove particles to control resolution is highly desirable when applying SPH to practical problems in fluid mechanics. A framework method to achieve this has been presented, making special use of linearly consistent SPH. The method highlights key concepts such as the use of a reference smoothing length to redefine individual smoothing lengths, and the use of a particle volume defined using kernel summations. As a demonstration, adaptive particle distribution with SPH has been shown to improve accuracy with a similar number of particles in a shock tube simulation.

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